

# Symmetric products of surfaces and the cycle index

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## Abstract

We study some of the combinatorial structures related to the signature of  $G$ -symmetric products of (open) surfaces  $SP_G^m(M) = M^m/G$  where  $G \subset S_m$ . The attention is focused on the question what information about a surface  $M$  can be recovered from a symmetric product  $SP^n(M)$ . The problem is motivated in part by the study of locally Euclidean topological commutative  $(m+k, m)$ -groups, [16]. Emphasizing a combinatorial point of view we express the signature  $\text{Sign}(SP_G^m(M))$  in terms of the cycle index  $Z(G; \bar{x})$  of  $G$ , a polynomial which originally appeared in Pólya enumeration theory of graphs, trees, chemical structures etc. The computations are used to show that there exist punctured Riemann surfaces  $M_{g,k}, M_{g',k'}$  such that the manifolds  $SP^m(M_{g,k})$  and  $SP^m(M_{g',k'})$  are often not homeomorphic, although they always have the same homotopy type provided  $2g+k = 2g'+k'$  and  $k, k' \geq 1$ .

## 1 Introduction

The complex plane  $\mathbb{C}$ , the punctured plane  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and the elliptic curves are classical examples of surfaces that support a group structure. A natural generalization of the (commutative) group structure is the structure of a (commutative)  $(m+k, m)$ -group.

Let  $SP^n(X) := X^n/S_n$  be the symmetric product of  $X$  and let  $SP^p(X) \times SP^q(X) \rightarrow SP^{p+q}(X)$ ,  $(a, b) \mapsto c := a * b$  be the operation induced by concatenation of strings  $a \in SP^p(X)$  and  $b \in SP^q(X)$ . A **commutative**  $(m+k, m)$ -groupoid is a pair  $(X, \mu)$  where the “multiplication”  $\mu$  is an arbitrary map

$\mu : SP^{m+k}(X) \rightarrow SP^m(X)$ . The operation  $\mu$  is associative if for each  $c \in SP^{m+2k}(X)$  and each representation  $c = a * b$ , where  $a \in SP^{m+k}(X)$  and  $b \in SP^m(X)$ , the result  $\mu(\mu(a) * b)$  is always the same, i.e. independent from the particular choice of  $a$  and  $b$  in the representation  $c = a * b$ . A commutative and associative  $(m+k, m)$ -groupoid is a  $(m+k, m)$ -group if the equation  $\mu(x * a) = b$  has a solution  $x \in SP^m(X)$  for each  $a \in SP^k(X)$  and  $b \in SP^m(X)$ . Note that  $(2, 1)$ -groups are essentially the groups in the usual sense of the word. If  $X$  is a topological space then  $(X, \mu)$  is a topological  $(m+k, m)$ -group if it is a  $(m+k, m)$ -group and the map  $\mu : SP^{m+k}(X) \rightarrow SP^m(X)$  is continuous.

For the motivation and other information about commutative  $(m+k, m)$ -groups the reader is referred to [16], [15]. Surprisingly enough, the only known surfaces that support the structure of a  $(m+k, m)$ -group for  $(m+k, m) \neq (2, 1)$  are of the form  $\mathbb{C} \setminus A$  where  $A$  is a finite set. Moreover it was proved in [15], see also Theorem 6.1 in [16] that if  $(M, \mu)$  is a locally Euclidean, topological, commutative,  $(m+k, m)$ -group then  $M$  must be an orientable 2-manifold. Moreover, a 2-manifold that admits the structure of a commutative  $(m+k, m)$ -group satisfies a strong necessary condition that the symmetric power  $SP^m(M) := M^m/S_m$  is of the form  $\mathbb{R}^u \times (S^1)^v$ . In particular the signature  $\text{Sign}(SP^m(M))$  of  $M$  must be zero.

There is a conjecture [16] that the only examples of topological, commutative,  $(m+k, m)$ -groups are supported by surfaces of the form  $M = \mathbb{C} \setminus A$ . Corollary 1.3 and Proposition 1.4 support this conjecture, since they imply that all open surfaces of sufficiently high genus have a non-zero signature. However this conjecture serves also as a partial motivation for the following general questions which may be of some independent interest.

### Questions:

- (A) To what extent is the topology of a surface  $M$  determined by the topology of its symmetric product  $SP^m(M)$  for a given  $m$ ?
- (B) Are there examples of non-homeomorphic (open) surfaces  $M$  and  $N$  such that the associated symmetric products  $SP^m(M)$  and  $SP^m(N)$  are homeomorphic?

In response to (A) we prove the following theorem which says that homological invariants alone are not sufficient to distinguish symmetric products of non-homeomorphic surfaces. This puts some limitations on surfaces  $M$  and  $N$  in question (B), however the question itself remains open and interesting already in the case of general surfaces  $M_{g,k}$ .

**Theorem 1.1** *There exist open, orientable surfaces  $M$  and  $N$  such that the associated symmetric products  $SP^m(M)$  and  $SP^m(N)$  are not homeomorphic although they have the same homotopy type. More precisely, this is always true if  $M = M_{g,k}$  and  $N = M_{g',k'}$  ( $k, k' \geq 1$ ) and*

- $2g + k = 2g' + k'$ ,
- $g \neq g'$  and  $\max\{g, g'\} \geq m/2$

where  $M_{g,k} := M_g \setminus \{x_1, \dots, x_k\}$  is the surface of genus  $g$  punctured at  $k$  points.

A natural approach to questions (A) and (B) is to determine what information about  $M$  is hidden in  $SP^m(M)$ . If  $SP^m(M)$  is known, then the iterated symmetric product  $SP^k(SP^m(M))$  and its higher order analogs are also known. Since  $SP^k(SP^m(M)) \cong SP_G(M) := M^{mk}/G$  where  $G = S_k \wr S_m$  is the wreath product of groups, it is natural to consider general  $G$ -symmetric products  $SP_G(M) = M^N/G$  where  $G \subset S_N$  is an arbitrary subgroup of  $S_N$ . When we want to emphasize that  $G$  is a subgroup of  $S_N$  we write  $SP_G(M) = SP_G^N(M)$ .  $SP_G(M)$  is always a  $\mathbb{Q}$ -homology manifold and the signature  $\text{Sign}(SP_G(M))$  is well defined. Our central technical result is the following theorem.

**Theorem 1.2**

$$\text{Sign}(SP_G^m(M_{g,k})) = Z(G; 0, -2g, 0, -2g, \dots) \quad (1)$$

where  $G \subset S_m$  and  $Z(G; x_1, x_2, \dots, x_m)$  is the cycle index of  $G$ .

**Corollary 1.3**

$$\text{Sign}(SP^{2n}(M_{g,k})) = (-1)^n \binom{g}{n}. \quad (2)$$

Recall that the cycle index, [1], [11], [3], [14], is a fundamental polynomial which was originally defined by J.H. Redfield and independently by G. Pólya who recognized its central role in his celebrated enumeration theory of groups, graphs, trees, chemical compounds etc. The advantage of expressing the result in terms of the cycle index lies in the fact that the cycle index is a well studied object of enumerative combinatorics. One of highlights is Theorem 1.16, see [1] Section V, which describes a procedure how the cycle index  $Z(G; \vec{x})$  of the wreath product  $G = Q \wr H$  can be in a transparent and elegant way expressed in terms of cycle indices  $Z(Q; \vec{x}), Z(H; \vec{x})$  of  $Q$  and  $H$  respectively. A good illustration how these ideas can be applied is provided by the following proposition which itself can be seen as another corollary of Theorem 1.2.

**Proposition 1.4** *Suppose that  $m$  is odd and  $p$  an even integer. Then*

$$\text{Sign}(SP_{S_p \wr S_m}(M_{g,k})) = Z(S_p \wr S_m; 0, -2g, \dots, 0, -2g) = (-1)^{p/2} \binom{\frac{1}{2}(2g)}{\frac{1}{2}p}. \quad (3)$$

**Corollary 1.5** *If  $m$  is an odd integer and  $k \geq 1$  then*

$$\text{Sign}(SP^2(SP^m(M_{g,k}))) = -\frac{1}{2} \binom{2g}{m}. \quad (4)$$

For completeness we recall a remarkable formula of Don Zagier<sup>1</sup> obtained by the Atiyah-Singer  $G$ -signature theorem applied to  $(\sigma_m, M^m)$ , where  $\sigma : M^m \rightarrow M^m$  is the cyclic permutation.

**Theorem 1.6** (Zagier [17])

$$\begin{aligned} \text{Sign}(SP_G^m(M)) &= Z(G; \tau, \chi, \tau, \chi, \dots) \\ &= Z(G; \vec{x})|_{x_{2i}=\chi, x_{2i-1}=\tau} \end{aligned} \quad (5)$$

where  $\tau = \tau(M)$  and  $\chi = \chi(M)$  are respectively the signature and the Euler characteristic of a compact, oriented, even dimensional manifold without boundary.

Again the use of the cycle index is convenient. In the case  $G = S_k \wr S_m$  one obtains, along the lines of Proposition 1.4, the following result which for  $m = 1$  reduces to the formula of Hirzebruch, [8], [17]. By convention [14],  $[t^p]f(t)$  is the coefficient of  $t^p$  in the power series  $f(t)$ .

**Theorem 1.7**

$$\text{Sign}(SP_{S_k \wr S_m}(M)) = [t^k](1 - t^2)^{\frac{1}{2}(-1)^{m+1} \binom{-\chi}{m}} \left( \frac{1+t}{1-t} \right)^{\frac{1}{2} \text{Sign}(SP^m(M))} \quad (6)$$

## 1.1 Signature as a function of both $G$ and $g$

Our proof of Theorem 1.2 with minor modification yields a proof of the corresponding well known statement for closed surfaces  $M_g$ . This allows us to check our computations so we find it convenient to formulate and prove these two results as parts of a single statement.

**Theorem 1.8**

$$\begin{aligned} \text{(a)} \quad \text{Sign}(SP_G^m(M_g)) &= Z(G; 0, 2 - 2g, 0, 2 - 2g, \dots) \\ &= Z(G; \vec{x})|_{x_{2i}=2-2g, x_{2i-1}=0} \\ \text{(b)} \quad \text{Sign}(SP_G^m(M_{g,k})) &= Z(G; 0, -2g, 0, -2g, \dots) \\ &= Z(G; \vec{x})|_{x_{2i}=-2g, x_{2i-1}=0} \end{aligned} \quad (7)$$

where  $Z(G; x_1, \dots, x_m)$  is the cycle index of the permutation group  $G \subset S_m$ .

Before we commence the proof, let us recall some generalities about the  $g$ -signature of  $G$ -manifolds or vector spaces with  $G$ -invariant bilinear forms, [9], [4].

Let  $V$  be a vector space,  $B : V \times V \rightarrow \mathbb{C}$  a hermitian bilinear form on  $V$  and  $g : V \rightarrow V$  an endomorphism which preserves the form  $B$ ,  $B(gx, gy) = B(x, y)$  for all  $x, y \in V$ . Then  $V$  admits a  $g$ -invariant decomposition  $V \cong V^0 \oplus V^+ \oplus V^-$

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<sup>1</sup>Zagier did not originally express the result in terms of the cycle index  $Z(G; \vec{x})$ . His result says that  $\text{Sign}(\sigma_m, M^m)$  is either  $\tau(M)$  or  $\chi(M)$  depending on the parity of  $m$ .

such that  $B$  is positive definite on  $V^+$ , negative definite on  $V^-$  and zero on  $V^0$ . Then the  $g$  signature of the triple  $(V, B, g)$ , or for short the  $g$ -signature of  $V$  is defined by

$$\text{Sign}(g, V) = \text{Sign}(g, V, B) = \text{Trace}(g|V^+) - \text{Trace}(g|V^-)$$

A symmetric or skew-symmetric form  $B : V \times V \rightarrow \mathbb{R}$  defined on a real vector space  $V$  can be extended to a hermitian form  $\hat{B}$  on the complexified space  $V \otimes \mathbb{C}$  by the formula [4],

$$\hat{B}(x \otimes \alpha, y \otimes \beta) = \begin{cases} \alpha \bar{\beta} B(x, y), & B \text{ is symmetric} \\ i \alpha \bar{\beta} B(x, y), & B \text{ is skew-symmetric} \end{cases} \quad (8)$$

The associated  $g$ -signature is also denoted by  $\text{Sign}(g, V)$ . Finally, suppose that  $M^{2n}$  is a smooth, oriented manifold, with or without boundary, and let  $g : M^{2n} \rightarrow M^{2n}$  be an orientation preserving diffeomorphism of  $M$ . The intersection form  $B : H_n(M, \mathbb{Q}) \times H_n(M, \mathbb{Q}) \rightarrow \mathbb{Q}$  is symmetric if  $n$  is even, or skew-symmetric if  $n$  is odd, and the associated endomorphism  $\tilde{g} := H_n(g) : H_n(M) \rightarrow H_n(M)$  preserves both the intersection form  $B$  and its complexification. The  $g$ -signature of  $(\tilde{g}, V, B)$  is in this case denoted by  $\text{Sign}(g, M)$ . In particular, we observe that the usual signature  $\text{Sign}(M)$  can be interpreted as the  $g$ -signature  $\text{Sign}(\text{Id}, M)$  of the identity map  $\text{Id} : M \rightarrow M$ .

The following well known result, [6], [9], is of fundamental importance. Note that even the case of 0-dimensional manifolds (finite sets) is interesting, when this result reduces to an elementary lemma (Burnside lemma) which is a corner-stone of Pólya enumeration theory.

**Proposition 1.9** *Suppose that  $M^{2n}$  is a smooth, oriented manifold with a not necessarily free, orientation preserving action of a finite group  $G$  of diffeomorphism. More generally, it is sufficient to assume that  $M$  is a  $\mathbb{Q}$ -homology manifold. Then  $M/G$  is a  $\mathbb{Q}$ -homology manifold,  $\text{Sign}(M/G)$  is well defined and the following formula holds, [6], [9],*

$$\text{Sign}(M/G) = \frac{1}{|G|} \sum_{g \in G} \text{Sign}(g, M).$$

The following proposition is used in the proof of Theorem 1.8.

**Proposition 1.10** *Let  $V$  be a  $(2n)$ -dimensional complex vector space and let  $B : V \times V \rightarrow \mathbb{C}$  be a hermitian form. Suppose  $\omega : V \rightarrow V$  is an endomorphism preserving the form  $B$ , such that  $\omega^{2n} = 1$ , and for some  $v_0 \in V$ , the set  $\{v_0, \omega(v_0), \dots, \omega^{2n-1}(v_0)\}$  is a basis of  $V$ . Suppose that  $B(v_0, \omega^j(v_0)) \neq 0 \implies j = n$ . Then,*

$$\text{Sign}(\omega, V) = \left( \sum_{x^{2n}=1} x^{n+1} \right) \text{sign } B(v_0, \omega^n(v_0)) = \begin{cases} 0, & n \neq 1 \\ 2 \text{sign } B(v_0, \omega(v_0)), & n = 1 \end{cases} \quad (9)$$

**Proof:** Let  $x \in \{1, \epsilon, \epsilon^2, \dots, \epsilon^{2n-1}\}$  be a root of unity,  $\epsilon = e^{2\pi i/2n}$ . Let  $z_x := v_0 + x^{-1}\omega(v_0) + x^{-2}\omega^2(v_0) + \dots + x^{-(2n-1)}\omega^{2n-1}(v_0)$  be the eigenvector of  $\omega$  corresponding to the eigenvalue  $x$ . If  $x$  and  $y$  are different eigenvalues,  $B(z_x, z_y) = B(\omega(z_x), \omega(z_y)) = x\bar{y}B(z_x, z_y) = 0$ . Otherwise, since  $B(v_0, \omega^j(v_0))$  can be nonzero only for  $j = n$ , we have  $B(z_x, z_x) = 2nx^n B(v_0, \omega^n(v_0))$ . By a well known formula,

$$\text{Sign}(V, B) = \sum_{\lambda \in \text{Spec}(\omega)} \lambda \text{Sign}(V_\lambda, B_\lambda),$$

where  $V_\lambda$  is the eigenspace of  $\omega$  which corresponds to the eigenvalue  $\lambda$  and  $B_\lambda = B|_{V_\lambda}$  is the restriction of the form  $B$  on  $V_\lambda$ . In our case,

$$\text{Sign}(V, B) = \sum_{x^{2n}=1} x \text{sign} B(z_x, z_x) = \left( \sum_{x^{2n}=1} x^{n+1} \right) \text{sign} B(v_0, \omega^n(v_0))$$

and the equation (9) follows.

Our next step in the preparations for the proof of Theorem 1.8 is an explicit description of the intersection pairing  $B : H_m(M_g^m; \mathbb{Q}) \times H_m(M_g^m; \mathbb{Q}) \rightarrow \mathbb{Q}$ . Note that both sides of the equations (7) are zero if  $m$  is an odd number. So from here on, we focus our attention on the even case and assume that  $m = 2n$ .

Let us choose an orientation on  $M_g$  and let  $\mathbb{T} \in H_2(M_g; \mathbb{Q})$  be the associated fundamental class. Let  $a_1, b_1, \dots, a_g, b_g$  be a symplectic basis of  $H_1(M_g)$  so that  $a_i \cap a_i = b_j \cap b_j = 0$ ,  $a_i \cap b_j = 0$  for  $i \neq j$ , and  $a_i \cap b_i = \mathbb{I}$ , where  $\mathbb{I} \in H_0(M_g)$  is the generator. By Künneth formula,

$$H_{2n}(M_g^{2n}; \mathbb{Q}) \cong \bigotimes_{k_1+k_2+\dots+k_{2n}=2n} H_{k_i}(M_g; \mathbb{Q}).$$

From here one deduces that a convenient basis of  $H_{2n}(M_g^{2n}; \mathbb{Q})$  consists of all “words”  $w = w_1 w_2 \dots w_{2n} = w_1 \times w_2 \times \dots \times w_{2n}$  where  $w_k \in \{a_i\}_{i=1}^g \cup \{b_j\}_{j=1}^g \cup \{\mathbb{I}, \mathbb{T}\}$ . Note that for dimensional reasons, the number of occurrences of the letter  $\mathbb{I}$  in the word  $w$  is equal to the number of occurrences of the letter  $\mathbb{T}$ .

**Lemma 1.11** *Let  $w = w_1 w_2 \dots w_{2n}$  and  $w' = w'_1 w'_2 \dots w'_{2n}$  be two words, representing basic homology classes in  $H_{2n}(M_g^{2n})$ . Then,*

$$B(w, w') = \epsilon_{w, w'} \langle w_1, w'_1 \rangle \dots \langle w_{2n}, w'_{2n} \rangle \quad (10)$$

where  $\epsilon_{w, w'}$  is either  $+1$  or  $-1$ , while  $B(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$  are the intersection pairings on groups  $H_{2n}(M_g^{2n})$  and  $H_1(M_g)$  respectively.

**Proof:** Indeed,  $w = w_1 \times \dots \times w_{2n} = \hat{w}_1 \cap \dots \cap \hat{w}_{2n}$  where,  $\hat{w}_i = \mathbb{T} \times \dots \times w_i \times \dots \times \mathbb{T} \in H_*(M_g^{2n})$ . Hence,  $B(w_1 \times \dots \times w_{2n}, w'_1 \times \dots \times w'_{2n}) = (\hat{w}'_1 \cap \dots \cap \hat{w}'_{2n}) \cap (\hat{w}_1 \cap \dots \cap \hat{w}_{2n})$  and it is sufficient to remember that

$$a \cap b = (-1)^{\text{cd}(a) \text{cd}(b)} b \cap a$$

where  $\text{cd}(x)$  is the codimension of a class  $x$ .

Let us define an involution  $*$  :  $H_*(M_g; \mathbb{Q}) \rightarrow H_*(M_g; \mathbb{Q})$  by the formula

$$a_i^* = b_i \quad b_j^* = a_j \quad \mathbb{I}^* = \mathbb{T} \quad \mathbb{T}^* = \mathbb{I} \quad (11)$$

i.e. the involution  $*$  is up to sign, the Poincaré duality map. For a given word  $w = w_1 w_2 \dots w_{2n}$  let  $w^* = w_1^* w_2^* \dots w_{2n}^*$ . Note that the first part of the following proposition is just a reformulation of Lemma 1.11, while the second part gives a precise formula for the sign function  $\epsilon_{w, w'}$ .

**Proposition 1.12**

$$B(w, w') \neq 0 \implies w' = w^*$$

$$B(w, w^*) = (-1)^{\binom{\alpha(w)}{2}} \langle w_1, w_1^* \rangle \dots \langle w_{2n}, w_{2n}^* \rangle = (-1)^{\binom{\alpha(w)}{2} + \beta(w)}$$

where  $\beta(w)$  is the number of occurrences of letters  $b_1, \dots, b_g$  in  $w$ , while  $\alpha(w)$  is the number of occurrences of both  $a_i$  and  $b_j$  in  $w$ . If  $\mathbb{I}$  and  $\mathbb{T}$  do not appear in  $w$  whatsoever, then  $B(w, w^*) = +1$ .

Let  $\pi \in S_{2n}$  be a permutation and let  $\alpha(\pi) = 1^{\alpha_1} 2^{\alpha_2} \dots (2n)^{\alpha_{2n}}$  be the associated partition of  $[2n] = \{1, 2, \dots, 2n\}$ . In light of the well known equality  $\text{Sign}(g \times h, M \times N) = \text{Sign}(g, M) \text{Sign}(h, N)$  [4] [9],

$$\text{Sign}(\pi, (M_g)^{2n}) = \prod_{k=1}^{2n} \{\text{Sign}(C_k, (M_g)^k)\}^{\alpha_k} \quad (12)$$

where  $C_k : (M_g)^k \rightarrow (M_g)^k$  is a cyclic permutation of coordinates,

$$C_k(x_1, x_2, \dots, x_k) = (x_2, \dots, x_k, x_1).$$

**Proposition 1.13**

$$\text{Sign}(C_k, (M_g)^k) = \begin{cases} 2 - 2g, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

**Proof:** Let  $V = H_k(M_g) \otimes \mathbb{C}$  and let  $B : V \times V \rightarrow \mathbb{C}$  be the hermitian form obtained by formula (8) from the intersection form on  $(M_g)^k$ . Let  $\omega : V \rightarrow V$  be the map induced by  $C_k$ . As before,  $\omega : V \rightarrow V$  is a  $B$ -preserving endomorphism. If  $V$  admits a  $B$ -orthogonal decomposition of the form  $V = V_1 \oplus V_2$  where  $V_1$  and  $V_2$  are  $\omega$ -invariant subspaces, then

$$\text{Sign}(\omega, V) = \text{Sign}(\omega_1, V_1) + \text{Sign}(\omega_2, V_2) \quad (13)$$

where  $\omega_i := \omega|_{V_i}$  is the restriction of  $\omega$  on  $V_i$ . Given a word  $w = w_1 \dots w_k$ , let  $V_w = \text{span}\{w, \omega(w), \dots, \omega^{k-1}(w)\}$  be the minimal  $\omega$ -invariant subspace of  $V$  which contains vector  $w$ . Let  $W_w := V_w + V_{w^*}$ . Then by Proposition 1.12, for

any two words  $w$  and  $w'$ , the associated spaces  $W_w$  and  $W_{w'}$  are either identical or mutually orthogonal. By formula (13), it suffices to find  $\text{Sign}(\omega, W_w)$ . If  $V_w \neq V_{w^*}$  then  $\text{Sign}(\omega, W_w) = 0$ , so we assume that  $W_w = V_w$ . Note that  $\dim(V_w) = p$ , where  $p = p(w) := \min\{s \geq 1 \mid \omega^s(w) = \pm w\}$ . If  $k$  is odd, then  $p$ , being a divisor of  $k$ , must be also an odd number. This implies that  $w^* \neq \pm \omega^j(w)$  for all  $j$ , which means that the form  $B|_{V_w}$  is zero and  $\text{Sign}(\omega, V_w) = 0$ . It immediately follows that  $\text{Sign}(C_k, (M_g)^k) = 0$  if  $k$  is an odd number. If  $k = 2m$  is even, then a nonzero contribution from  $V_w$  can be expected only if  $p$  is an even number. In this case we can apply Proposition 1.10 and deduce that  $p = 2$ . The list of all basic words  $u$  having the property  $p(u) = 2$  is

$$w = \mathbb{I}\mathbb{T} \dots \mathbb{I}\mathbb{T}, \quad w_i := a_i b_i \dots a_i b_i, \quad i = 1, \dots, g.$$

Since  $\mathbb{I}$  and  $\mathbb{T}$  are classes of even degree,  $\omega(w) = w^*$  and  $B(w, \omega(w)) = +1$ . Since  $a_i, b_i$  are classes of degree 1,  $B(w_i, \omega(w_i)) = (-1)^{2m-1} B(w_i, w_i^*)$ , and knowing that by Proposition 1.12,  $B(w_i, w_i^*) = (-1)^{\beta(w_i)+m(2m-1)} = +1$ , we have  $B(w_i, \omega(w_i)) = -1$ . Finally, by Proposition 1.10,

$$\text{Sign}(C_{2m}, (M_g)^{2m}) = \text{Sign}(V_w) + \sum_{i=1}^g \text{Sign}(V_{w_i}) = 2 - 2g.$$

**Proposition 1.14** *If  $s \geq 1$  then*

$$\text{Sign}(C_k, (M_{g,s})^k) = \begin{cases} -2g, & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$$

**Proof:** As in the proof of Proposition 1.13, we need information about the intersection pairing

$$B : H_{2n}(M_{g,k}^{2n}; \mathbb{Q}) \times H_{2n}(M_{g,k}^{2n}; \mathbb{Q}) \rightarrow \mathbb{Q}.$$

The homology group  $H_1(M_{g,s}, \mathbb{Z}) \cong \mathbb{Z}^{2g+s-1}$  has a basis  $e_1, e_2, \dots, e_{2g+s-1}$  where  $e_i = a_i, e_{g+j} = b_j$  for  $i = 1, \dots, g$  and  $e_j$  for  $j \geq 2g+1$  correspond to the holes  $\alpha_1, \dots, \alpha_{s-1}$  in  $M_g$ . The group  $H_{2n}(M_{g,s}^{2n}; \mathbb{Q})$  is generated by the classes (words) of the form  $w = w_1 \times \dots \times w_{2n} = w_1 \dots w_{2n}$  where  $w_i \in \{e_1, \dots, e_{2g+s-1}\}$ . Lemma 1.11 is still true with the simplification that both  $\mathbb{I}, \mathbb{T}$  and the classes  $\{e_j\}_{j=2g+1}^{2g+s-1}$ , associated to the holes in  $M_{g,s}$ , are excluded. The rest of the proof follows the argument of the proof of Proposition 1.13.

**Proof of Theorem 1.8:** As it was already observed, Theorem 1.8 is trivially correct if  $m$  is an odd number since in that case both sides of the equation (7) are zero. Let us assume that  $m$  is an even number,  $m = 2n$ . By equation (12) and Proposition 1.13

$$\begin{aligned} \text{Sign}(SP_G^{2n}(M_g)) &= \frac{1}{(2n)!} \sum_{\pi \in G} \text{Sign}(\pi, (M_g)^{2n}) \\ &= \frac{1}{(2n)!} \sum_{\pi \in G} \prod_{k=1}^{2n} \{\text{Sign}(C_k, (M_g)^k)\}^{\alpha_k(\pi)} \\ &= Z(G; 0, 2-2g, \dots, 0, 2-2g) \end{aligned} \quad (14)$$



which establishes part (a) of Theorem 1.8. The part (b) follows by the same argument from Proposition 1.14.

## 1.2 Corollaries of Theorem 1.2

We start with the following elementary lemma.

**Lemma 1.15**

$$Z(S_n; \alpha, \beta, \alpha, \beta, \dots) = [t^n](1-t)^{-\frac{1}{2}(\alpha+\beta)}(1+t)^{\frac{1}{2}(\alpha-\beta)} \quad (15)$$

**Proof:** The result is easily deduced from the well known fact [1], [11], [3], [14]

$$Z(S_n; x_1, \dots, x_n) = [t^n] \exp(x_1 t + \frac{x_2 t^2}{2} + \frac{x_3 t^3}{3} + \dots). \quad (16)$$

In particular for  $\alpha = \beta$  one has

$$Z(S_n; \alpha, \alpha, \dots) = [t^n](1-t)^{-\alpha} = (-1)^n \binom{-\alpha}{n} \quad (17)$$

while for  $\alpha = 0$  formula (15) reduces to

$$Z(S_n; 0, \beta, 0, \beta, \dots) = [t^n](1-t^2)^{-\beta/2}. \quad (18)$$

**Theorem 1.16** (G. Pólya, [1]) *Let  $G = S_k \wr H$  be the wreath product of  $S_k$  by a subgroup  $H \subset S_m$  of the symmetric group  $S_m$ . Then the cycle index  $Z(G; \vec{x})$  of  $G$  can be computed from the cycle indices of  $S_k$  and  $H$  by the formula*

$$Z(G; \vec{x}) = Z(S_k; Z(H; x_1, x_2, \dots), Z(H; x_2, x_4, \dots), \dots, Z(H; x_k, x_{2k}, \dots)). \quad (19)$$

**Proof of Proposition 1.4:** By Theorems 1.2 and 1.16

$$\begin{aligned} \text{Sign}(SP_{S_p \wr S_m}(M_{g,k})) &= Z(S_p \wr S_m; 0, -2g, 0, -2g, \dots) \\ &= Z(S_p; Z(S_m; 0, -2g, 0, -2g, \dots), Z(S_m; -2g, -2g, \dots), \dots) \\ &= Z(S_p; 0, (-1)^m \binom{2g}{m}, 0, (-1)^m \binom{2g}{m}, \dots) = (-1)^{p/2} \binom{\frac{1}{2}(2g)}{\frac{1}{2}p}. \end{aligned}$$

## 1.3 Non-homeomorphic symmetric products

**Proof of Theorem 1.1:** Since  $M_{g,k}$  is, up to homotopy, a wedge of  $2g + k - 1$  circles, the condition  $2g + k = 2g' + k'$  implies  $M_{g,k} \simeq M_{g',k'}$  and as a consequence  $SP^m(M_{g,k}) \simeq SP^m(M_{g',k'})$ . Suppose that  $m$  is even integer,  $m = 2n$ . The open manifold  $SP^{2n}(M_{g,k})$ , according to Corollary 1.3, has signature  $(-1)^n \binom{g}{n}$ . The condition  $\max\{g, g'\} \geq n$  guarantees that either  $\binom{g}{n}$  or  $\binom{g'}{n}$  is nonzero. The

sequence  $\binom{g}{n}$ , as a function of  $g$ , is strictly monotone for  $g \geq n$ . Together with the condition  $g \neq g'$  this implies

$$\text{Sign}(SP^{2n}(M_{g,k})) \neq \text{Sign}(SP^{2n}(M_{g',k'})),$$

hence  $SP^{2n}(M_{g,k})$  and  $SP^{2n}(M_{g',k'})$  are not homeomorphic. The case of an odd integer  $m$  is treated similarly. If contrary to the claim  $SP^m(M_{g,k})$  and  $SP^m(M_{g',k'})$  are homeomorphic, then by Corollary 1.5  $\binom{2g}{m} = \binom{2g'}{m}$ . This again would contradict the conditions  $\max\{g, g'\} \geq m/2$  and  $g \neq g'$ .

## References

- [1] M. Aigner. *Combinatorial Theory*, Springer-Verlag, Berlin 1979.
- [2] A. Björner. Topological methods. In R. Graham, M. Grötschel, and L. Lovász, editors. *Handbook of Combinatorics*. North-Holland, Amsterdam, 1995.
- [3] I.M. Gessel, R.P. Stanley. Algebraic enumeration. In *Handbook of Combinatorics II*, R.L. Grakham, M. Grötschel, L. Lovász (eds.), North-Holland, Amsterdam 1995.
- [4] C.M. Gordon. The  $G$ -signature theorem in dimension 4, in *A la Recherche de la Topologie Perdue*, L. Guillou, A. Marin (eds.), Progress in Math. 62, Birkhäuser 1986.
- [5] P. Griffiths, J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, 1978.
- [6] A. Grothendieck. Sur quelques points d’algebre homologique. *Tôhoku Math. J.*, (2) 9 (1957), 119–221.
- [7] F. Hirzebruch. The signature theorem: reminiscences and recreation. in *Prospects in mathematics*, Ann. of math. Studies 70, Princeton Univ. Press 1971, 3–31.
- [8] F. Hirzebruch. *Lectures on the Atiyah-Singer theorem and its applications*. (unpublished lecture notes), Berkeley 1968.
- [9] F. Hirzebruch, D. Zagier. *The Atiyah – Singer Theorem and Elementary Number Theory*. Publish or Perish, Inc., Boston, 1974.
- [10] S. Kallel. Divisor spaces on punctured Riemann surfaces. *Trans. Amer. Math. Soc.*, 350 (1998), 135–164.
- [11] V. Krishnamurthy. *Combinatorics, Theory and Applications*, East–West Press 1985.
- [12] I.G. Macdonald. The Poincaré polynomial of a symmetric product. *Proc. Camb. Phil. Soc.*, 1962.
- [13] I.G. Macdonald. Symmetric product of an algebraic curve. *Topology* 1 (1962), 319–343.
- [14] R. Stanley *Enumerative Combinatorics*, Vol. I and II. Stud. Advanced Math. No. 49, Cambridge Univ. Press 2001.
- [15] K. Trenčevski, D. Dimovski. *Complex Commutative Vector Valued Groups*. Macedonian Acad. Sci. and Arts, Skopje, 1992.
- [16] K. Trenčevski, D. Dimovski. On the affine and projective commutative  $(m+k, m)$ -groups. *J. of Algebra*, 240 (2001), 338–365.
- [17] D.B. Zagier. *Equivariant Pontrjagin Classes and Applications to Orbit Spaces*. Lecture Notes in Mathematics 290, Springer-Verlag 1972.